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Three-body choreographies in given curves

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Abstract
As shown by Johannes Kepler in 1609, in the two-body problem, the shape of the orbit, a given ellipse, and a given non-vanishing constant angular momentum determine the motion of the planet completely. Even in the three-body problem, in some cases, the shape of the orbit, conservation of the center of mass and a constant of motion (the angular momentum or the total energy) determine the motion of the three bodies. We show, by a geometrical method, that choreographic motions, in which equal mass three bodies chase each other around the same curve, will be uniquely determined for the following two cases. (i) Convex curves that have point symmetry and non-vanishing angular momentum are given. (ii) Eight-shaped curves which are similar to the curve for the figure-eight solution and the energy constant are given. The reality of the motion should be tested whether the motion satisfies an equation of motion or not. Extensions of the method for generic curves are shown. The extended methods are applicable to generic curves which do not have point symmetry. Each body may have its own curve and its own non-vanishing masses.

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1. Introduction

The three-body figure-eight solution is one of the solutions of the planar equal mass three-body problem under the Newtonian gravity. In this solution, three bodies chase each other around a fixed eight-shaped curve. It was found numerically by Moore [1] and its existence was proved by Chenciner and Montgomery [2].

Only a little is known about the eight-shaped curve. Simó showed numerically that the curve cannot be expressed by algebraic curves of orders 4, 6, 8 [3, 4]. Chenciner and Montgomery [2] showed that the curve is a 'star shape', namely a ray from the origin meets the curve at most once. Fujiwara and Montgomery [5] proved that each lobe of the eight-shaped curve is convex.

On the other hand, the present authors found a parameterization $q_i(t) = (x_i(t), y_i(t))$ of the lemniscate of Bernoulli $(x^2 + y^2)^2 = x^2 - y^2$, which satisfies an equation of motion under an inhomogeneous potential [6]:

$$\frac{d^2 q_i}{dt^2} = -\frac{\partial V}{\partial q_i}, \tag{1}$$

$$V = \sum_{i<j} \left(\frac{1}{2} \ln r_{ij} - \frac{\sqrt{3}}{24} r_{ij}^2 \right), \tag{2}$$

where $r_{ij} = |q_i - q_j|$ is the mutual distance between the body i and j.

An interesting point of their approach is that they started the arguments from the lemniscate curve, without any assumption for the potential. They showed that there is a parameterization $q_i(t)$ of the curve that keeps the geometric center of mass being at the origin $\sum_i q_i(t) = 0$ and keeps the angular momentum being zero $\sum q_i(t) \times dq_i(t)/dt = 0$ for all t. Using this parameterization, they searched for what kind of potential can support this motion. Finally, they found the potential (2).

Then a question arises. Does a similar approach work for other eight-shaped curves? Namely, can we determine the three-body motion $q_i(t)$ if the shape of the orbit for the figure-eight solution is known?

This approach works for the two-body problem as shown by Johannes Kepler in 'Astronomia Nova' published in 1609. In his book, he stated the first law, planets move in the elliptical curve with the sun at one focus. Then his second law, which is now known as the conservation of the angular momentum, determines the motion of a planet in the ellipse if non-vanishing constant angular momentum is given.

In this paper, we show that this approach works in the three-body problem. Namely, for some curves, conservation of the center of mass and a constant of the motion (the angular momentum or the total energy) determine the three-body motion. Here, in the two-body problem, the total energy constant, instead of vanishing angular momentum, determines the motion if the orbit is linear.

To demonstrate the main idea, let us observe how the condition for geometrical center of mass being at the origin,

$$q_1 + q_2 + q_3 = 0, \tag{3}$$

determines the mutual positions of the three bodies. Let us consider a unit circle $|z| = 1$ in the complex plane and a position $q_3 = \exp(i\phi)$ on the circle. We know that the set of two points $\{q_1, q_2\}$ on the same circle that satisfy $q_1 + q_2 + q_3 = 0$ is $\{q_1, q_2\} = \{\exp(i(\phi + 2\pi/3)), \exp(i(\phi - 2\pi/3))\}$. On the other hand, it is obvious that the two points are the cross points of the original circle $|z| = 1$ and the unit circle $|z + q_3| = 1$, which is the parallel translation $z \mapsto z - q_3$ of the original circle.

One of the authors, Ozaki, noted that this is not an accident. He found the following theorem.

Theorem 1 (Construction of three points)**.** *If a curve γ in $\mathbb{R}^d$ with $d = 2, 3, 4, \ldots$ is invariant under the inversion $q \mapsto -q$, then the set $\{\{q_1, q_2\} | q_1, q_2 \in \gamma, q_1 + q_2 + q_3 = 0\}$ for a given $q_3 \in \gamma$ is equal to the set $\{\{q, q^*\} | q \in \gamma \cap \gamma_\parallel\}$ where $\gamma_\parallel$ is the parallel translation $q \mapsto q - q_3$ of the curve γ and $q^* = -q - q_3$.*

This theorem gives us a method to find three points q_1, q_2, q_3 with $q_1 + q_2 + q_3 = 0$ on a point symmetric curve with respect to the origin. This theorem states that for such a curve and for a given $q_3 \in \gamma$, (i) if there is a pair $q_1, q_2 \in \gamma$ that satisfies $q_1 + q_2 + q_3 = 0$, then the points q_1 and q_2 should be the cross points of γ and $\gamma_\parallel$, and inversely, (ii) if a cross point q of γ and $\gamma_\parallel$ exists, then the point $q^* = -q - q_3$ is also a cross point of the same curves, and $q + q^* + q_3 = 0$ is satisfied.

See figures 2, 4, 6 and 7. Figure 2 shows the situation for a convex curve that is invariant under $q \mapsto -q$. This figure suggests that the pair of the cross points $\{q_1, q_2\} = \{q, q^*\}$, namely the solution of $q_1 + q_2 + q_3 = 0$, is unique for $q_3 \in \gamma$. Figures 4, 6 and 7 show the situation for an eight-shaped curve. These figures suggest that there are two pairs of the cross points: trivial pair $\{O, -q_3\}$, in which three points $-q_3$, O and q_3 are collinear and one non-trivial pair $\{q_1, q_2\}$.

For these cases, we can show that the (non-trivial) pair $\{q_1, q_2\}$ is determined uniquely for a given $q_3 \in \gamma$. Moreover, we can show that if we move q_3 around the whole curve, the points q_1 and q_2 move smoothly and strongly monotonically around the whole curve without collisions. Thus, the motion around such a curve is determined uniquely modulo time re-parameterization, $q_i(t) \mapsto q_i(\tau(t))$ with a function $\tau(t)$.

In section 2, proofs of theorem 1 are given. In theorem 1, the points q_1 and q_2 are assumed to be constrained on a same curve, the curve is assumed to be point symmetric and the three bodies are assumed to have the same mass. We can remove there assumptions. Extensions of theorem 1 are also given in section 2. In section 3, considering the geometrical property of the cross points of the convex curve and its translation we prove the uniqueness and the smoothness of $\{q_1, q_2\}$ for a given q_3. Then we show that motions of equal mass three bodies in planar point symmetric convex curves with respect to the origin are uniquely determined if non-vanishing angular momentum is given. In section 4, we show, the main result, the uniqueness of the motions of equal mass three bodies in planar eight-shaped curves if the energy constant is given. Section 5 presents the summary and discussions.

2. Constructions of three points

In this section, we prove some geometrical constructions of three points whose geometrical center of mass is fixed to the origin, namely theorem 1 and its extensions.

Theorem 1 is a corollary of the following more general theorem.

Theorem 2. *For a given set $\gamma_1, \gamma_2 \subset \mathbb{R}^d$ and $q_3 \in \mathbb{R}^d$, we have the following equalities:*

$$\{\{q_1, q_2\} | q_1 \in \gamma_1, q_2 \in \gamma_2, q_1 + q_2 + q_3 = 0\} = \{\{q_1, q_1^*\} | q_1 \in \gamma_1 \cap \gamma_2^*\} \tag{4}$$

$$= \{\{q_2, q_2^*\} | q_2 \in \gamma_1^* \cap \gamma_2\}, \tag{5}$$

where * represents a map $q \mapsto q^* = -q - q_3$ and γ^* is the image of γ by this map.

Proof of theorem 2. We prove equation (4). The proof for equation (5) is similar. If q_1 and q_2 satisfy $q_1 \in \gamma_1, q_2 \in \gamma_2$ and $q_1 + q_2 + q_3 = 0$, then $q_1 = -q_2 - q_3 = q_2^* \in \gamma_2^*$. Therefore, $q_1 \in \gamma_1 \cap \gamma_2^*$ and $q_2 = q_1^*$. Inversely, if q_1 and q_2 are given by $q_1 \in \gamma_1 \cap \gamma_2^*$ and $q_2 = q_1^*$, then $q_1 \in \gamma_1 \cap \gamma_2^* \subset \gamma_1$ and $q_2 = q_1^* \in \gamma_1^* \cap \gamma_2 \subset \gamma_2$. Moreover, by the definition of $q_2 = q_1^*$ we get $q_1 + q_2 + q_3 = 0$. □

If q_1, q_2 and q_3 move around the same set γ, we have the following corollary by simply making $\gamma_1 = \gamma_2 = \gamma$.

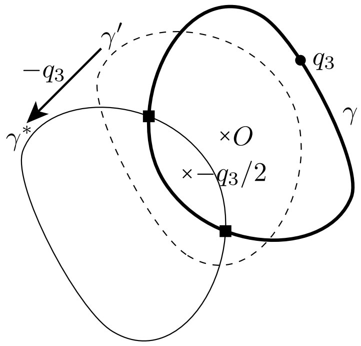

Figure 1. The curve γ^* that is the image of γ by the map $q \mapsto q^* = -q - q_3$ can be drawn by two ways. (i) Make the inversion of γ with respect to the point $-q_3/2$. (ii) Draw the inversion $q \mapsto -q$ of γ with respect to the origin to get γ'. Then make parallel translation $q \mapsto q - q_3$ of γ'.

Corollary 3. *For a given set $\gamma \subset \mathbb{R}^d$ and $q_3 \in \gamma$, we have the following equality:*

$$\{\{q_1, q_2\} | q_1, q_2 \in \gamma, q_1 + q_2 + q_3 = 0\} = \{\{q, q^*\} | q \in \gamma \cap \gamma^*\}. \tag{6}$$

We do not assume any symmetry for the set γ in this corollary. So, this corollary can be used to make equal mass three-body motions in given curves with no symmetry.

Proof of theorem 1. Note that the map $q \mapsto q^* = -q - q_3$ can be decomposed into the map $q \mapsto -q$ followed by the map $q \mapsto q - q_3$. Therefore, the curve γ^* can be made by the two steps. First make inversion γ' of γ by $q \mapsto -q$, then make parallel translation $\gamma'_{\parallel}$ of γ' by $q \mapsto q - q_3$. See figure 1. For the case when γ is invariant under the inversion, $q \mapsto -q$; then $\gamma' = \gamma$ and $\gamma^* = \gamma_{\parallel}$. Thus, we get a proof of theorem 1. □

Remark for corollary 3 and theorem 1. In the above proof, the condition $q_3 \in \gamma$ is not used. Actually, corollary 3 and theorem 1 are true for $q_3 \in \mathbb{R}^d$.

Remark for theorem 2. For three bodies with general masses $m_i \neq 0$, the center of mass being at the origin is defined by

$$\sum_{i=1,2,3} m_i q_i = 0. \tag{7}$$

For this case, let

$$\tilde{q}_i = m_i q_i \tag{8}$$

and $\tilde{\gamma}_i$ be the image of the curves γ_i by the map $q_i \mapsto \tilde{q}_i = m_i q_i$. Then the conditions $q_1 \in \gamma_1, q_2 \in \gamma_2$ and equation (7) are equivalent to $\tilde{q}_1 \in \tilde{\gamma}_1, \tilde{q}_2 \in \tilde{\gamma}_2$ and

$$\sum_{i=1,2,3} \tilde{q}_i = 0, \tag{9}$$

respectively. Then we can apply theorem 2 for $\tilde{q}_i$ and $\tilde{\gamma}_i$. Once we find the positions $\tilde{q}_1$ and $\tilde{q}_2$, we get the positions $q_i = m_i^{-1} \tilde{q}_i$ for $i = 1, 2$.

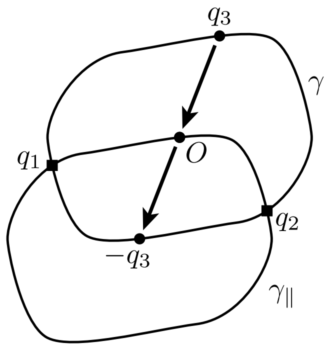

Figure 2. The closed convex curve γ is point symmetric with respect to the origin O. For a given point $q_3 \in \gamma$, the two points on γ that satisfy $q_1 + q_2 + q_3 = 0$ are given by the cross points of γ and $\gamma_{\parallel}$.

3. Three-body choreography in a point symmetric convex curve

3.1. Motion in a point symmetric convex curve

In this subsection, as a simple application of theorem 1, we construct an equal mass three-body motion in a given closed convex curve γ that is invariant under the inversion $q \mapsto -q$. We assume that the curvature is not zero everywhere on γ.

Theorem 4. *If a closed planar convex curve γ is invariant under the inversion $q \mapsto -q$ and its curvature is not zero everywhere on γ, the solutions of $q_1 + q_2 + q_3 = 0$ with $q_1, q_2 \in \gamma$ for a given $q_3 \in \gamma$ are unique. Moreover, when q_3 moves around γ, the motion $q_i(\sigma)$, $i = 1, 2$, are smooth, i.e. $|\mathrm{d}q_i/\mathrm{d}\sigma| < \infty$, and strongly monotonic, i.e. $\mathrm{d}q_i/\mathrm{d}\sigma \neq 0$, where σ is the curve length for q_3.*

For a given $q_3 \in \gamma$, the pair of positions $\{q_1, q_2\}$ is given by theorem 1 as $\{q_1, q_2\} = \{q, q^*\}$. First, we show the uniqueness of the pair $\{q_1, q_2\}$. As shown in figure 2, the map $q \mapsto q - q_3$ maps $q_3 \in \gamma$ to the origin O, O to the point $-q_3 \in \gamma - \gamma_{\parallel}$ and $-q_3$ to the $-2q_3 \in \gamma_{\parallel} - \gamma$. Therefore, the curve $\gamma_{\parallel}$ starts at the origin O which is surrounded by γ and passes the point $-2q_3$ which is outside of γ. Then there are at least two points in $\gamma \cap \gamma_{\parallel}$. On the other hand, $\gamma \cap \gamma_{\parallel}$ has at most two elements by lemma 2 in appendix A. Therefore, $\gamma \cap \gamma_{\parallel}$ has exactly two elements. Thus, we find a unique pair $\{q_1, q_2\} = \{q, q^*\}$ that satisfies $q_1 + q_2 + q_3 = 0$ by theorem 1.

Let us move the point q_3 around the whole curve to one direction, namely using the curve length σ for q_3,

$$q_3 = q_3(\sigma) \qquad \text{with} \quad \left|\frac{\mathrm{d}q_3}{\mathrm{d}\sigma}\right| = 1. \tag{10}$$

Then q_1 and q_2 are uniquely parameterized by the same parameter σ. To prove that $q_i(\sigma)$ for $i = 1, 2$ are smooth and strongly monotonic functions of σ, i.e.

$$\left|\frac{\mathrm{d}q_i(\sigma)}{\mathrm{d}\sigma}\right| < \infty \qquad \text{and} \qquad \frac{\mathrm{d}q_i(\sigma)}{\mathrm{d}\sigma} \neq 0, \tag{11}$$

note that when q_3 moves around γ with some speed, $\gamma_{\parallel}$ moves to the opposite direction with the same speed since the center of $\gamma_{\parallel}$ is $-q_3$. Therefore,

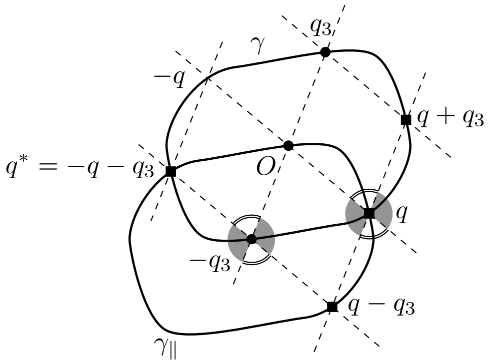

Figure 3. Lemma 1 for the closed convex curve with point symmetry. We denote one of the cross points of γ and $\gamma_{\|}$ by q. (i) The parallelogram $\alpha q + \beta q_3$, $0 \leqslant \alpha, \beta \leqslant 1$, is included inside γ; thus, at q, the tangent line to the curve γ passes through the shaded area, while the tangent line to the curve $\gamma_{\|}$ at q passes in the non-shaded area because the parallelogram $\alpha q - \beta q_3$, $0 \leqslant \alpha, \beta \leqslant 1$, is included inside the $\gamma_{\|}$. Therefore, the tangent lines at q to the lines γ and $\gamma_{\|}$ are distinct. (ii) The parallelogram $-\alpha q - \beta q_3$, $0 \leqslant \alpha, \beta \leqslant 1$, is included inside γ; thus the tangent line to the curve γ at $-q_3$ passes through the shaded area. Therefore, the tangent line to $\gamma_{\|}$ at q and the tangent line to γ at $-q_3$ are not parallel.

Lemma 1. *In theorem 1, (i) if the tangent lines to the curves γ and $\gamma_{\|}$ at q are distinct, then $q(\sigma)$ and $q^*(\sigma)$ are smooth functions of σ where σ is the curve length for q_3. (ii) Further, if the tangent line to the curve γ at $-q_3$ is not parallel to the tangent line to the curve $\gamma_{\|}$ at q, then $q(\sigma)$ and $q^*(\sigma)$ are smooth and strongly monotonic function of σ.*

For γ in theorem 4, from figure 3 it is clear that both conditions (i) and (ii) are satisfied for all $q_3 \in \gamma$; therefore, $q_1(\sigma)$ and $q_2(\sigma)$ are smooth and strongly monotonic by lemma 1. Then theorem 4 is proved.

Now we demand the angular momentum

$$c = \sum_{i=1,2,3} q_i(\sigma(t)) \times \frac{\mathrm{d}q_i(\sigma(t))}{\mathrm{d}t} = \frac{\mathrm{d}\sigma}{\mathrm{d}t} \sum_{i=1,2,3} q_i(\sigma) \times \frac{\mathrm{d}q_i(\sigma)}{\mathrm{d}\sigma} \tag{12}$$

to be constant in order to investigate the motion in γ. For the convex curve, we have

$$J(\sigma) = \sum_{i=1,2,3} q_i(\sigma) \times \frac{\mathrm{d}q_i(\sigma)}{\mathrm{d}\sigma} \neq 0. \tag{13}$$

This is because, if $J(\sigma) = 0$, three tangent lines must meet at a point by the three tangents theorem found by the present authors [5–7], whereas there are at most two tangent lines to the convex curve from one point. Thus, the time dependence of σ is determined by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{c}{J(\sigma)}. \tag{14}$$

Accordingly, the motion $q_i(\sigma(t))$ is determined uniquely by c.

Since $J(\sigma) \neq 0$, the sign of $\mathrm{d}\sigma/\mathrm{d}t$ is fixed. Let us take the sign of c be positive, then the points $q_i(\sigma(t))$ move anti-clockwise. By the uniqueness of $\{q_1, q_2\}$ if q_3 moves around the whole γ, the other points q_1 and q_2 move around the whole curve without collision. So, we can name the three points q_1, q_2 and q_3 in anti-clockwise order.

In the following, we write $q_i(t) = q_i(\sigma(t))$ for simplicity. At time $t = 0$, the three points are at $q_i(0)$. As time passes, the point $q_1(t)$ moves around the curve toward the point $q_2(0)$, and at some time t_0, $q_1(t)$ reaches to the point $q_2(0)$. Then we have

$$q_1(t_0) = q_2(0), \qquad q_2(t_0) = q_3(0), \qquad q_3(t_0) = q_1(0), \tag{15}$$

because one position determines the other two positions uniquely. Then the motion for $t_0 \leqslant t \leqslant 2t_0$ is also determined by the motion $q_i(t)$ for $0 \leqslant t \leqslant t_0$ as follows,

$$q_1(t) = q_2(t - t_0), \qquad q_2(t) = q_3(t - t_0), \qquad q_3(t) = q_1(t - t_0), \tag{16}$$

because again one position determines the other two positions uniquely and the name shifts $1 \to 2$, $2 \to 3$ and $3 \to 1$ are equivalent to the time shift $t \to t + t_0$.

We can proceed the same step over and over again; therefore, we have a periodic motion of $q_i(t)$ with the period $T = 3t_0$ and the motion is described by

$$q_1(t) = q_1(t), \qquad q_2(t) = q_1(t + T/3), \qquad q_3(t) = q_1(t + 2T/3). \tag{17}$$

We would like to call this periodic motion with an equal time spacing a 'choreography' in the point symmetric convex curve. We should note that there is no guarantee for this motion to satisfy some equation of motion.

3.2. Three-body choreography in an ellipse

In this subsection, let us assume the curve γ is an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{18}$$

with constants $a, b > 0$. This is convex and invariant under the inversion with respect to the origin, $(x, y) \mapsto (-x, -y)$. Therefore, by theorem 4, three-body choreography in this curve satisfying $q_1 + q_2 + q_3 = 0$ with constant angular momentum c is determined uniquely. For this case, we can construct a choreography explicitly and show that this choreography satisfies the equation of motion for harmonic oscillators.

This ellipse is parameterized by

$$q(\tau) = (x(\tau), y(\tau)) = (a\cos(\tau), b\sin(\tau)) \tag{19}$$

with an arbitrary parameter τ. Then the points

$$q_1(\tau) = q(\tau), \qquad q_2(\tau) = q(\tau + 2\pi/3), \qquad q_3(\tau) = q(\tau + 4\pi/3) \tag{20}$$

satisfy $q_1 + q_2 + q_3 = 0$. Since

$$q(t) \times \frac{\mathrm{d}q(\tau)}{\mathrm{d}\tau} = ab, \tag{21}$$

we get

$$\sum q_i(\tau) \times \frac{\mathrm{d}q_i(\tau)}{\mathrm{d}\tau} = 3ab. \tag{22}$$

Therefore, the equation $c = \mathrm{d}\tau/\mathrm{dt} \sum \mathrm{q_i}(\tau) \times \mathrm{dq_i}(\tau)/\mathrm{d}\tau$ determines

$$\frac{\mathrm{d}\tau}{\mathrm{dt}} = \frac{c}{3ab}. \tag{23}$$

Thus, the three-body choreography in the ellipse is uniquely determined by

$$q(t) = (a\cos(\omega t), b\sin(\omega t)), \qquad \text{with} \quad \omega = \frac{c}{3ab} \tag{24}$$

and

$$q_1(t) = q(t), \qquad q_2(t) = q\left(t + \frac{2\pi}{3\omega}\right), \qquad q_3(t) = q\left(t + \frac{4\pi}{3\omega}\right) \tag{25}$$

Obviously, $q_i(t)$ satisfies the equation of motion for the harmonic oscillator

$$\frac{\mathrm{d}^2 \mathrm{q_i(t)}}{\mathrm{dt}^2} = -\omega^2 q_i(t). \tag{26}$$

Using $q_1 + q_2 + q_3 = 0$, we get $q_i = \sum_{j \neq i}(q_i - q_j)/3$. Therefore, q_i satisfy the following equations of motion:

$$\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = \frac{\omega^2}{3}\sum_{j \neq i}(q_j - q_i) = -\frac{\partial V}{\partial q_i}, \tag{27}$$

with

$$V = \frac{\omega^2}{6}\sum_{i \neq j}|q_i - q_j|^2. \tag{28}$$

Therefore, the choreography in an ellipse is realized by the Hamiltonian

$$H = \sum_i \frac{|p_i|^2}{2} + \frac{\omega^2}{6}\sum_{i \neq j}|q_i - q_j|^2. \tag{29}$$

For this choreography, the kinetic energy K, the potential energy V and the moment of inertia I are the following constants:

$$K = \frac{1}{2}\sum_i \left|\frac{\mathrm{d}q_i}{\mathrm{d}t}\right|^2 = \frac{3\omega^2}{4}(a^2 + b^2), \tag{30}$$

$$V = \frac{\omega^2}{6}\sum_{i \neq j}|q_i - q_j|^2 = \frac{3\omega^2}{4}(a^2 + b^2), \tag{31}$$

$$I = \sum_i |q_i|^2 = \frac{3}{2}(a^2 + b^2), \tag{32}$$

respectively.

4. Three-body choreography in an eight-shaped curve

We consider the eight-shaped curve γ defined by the following properties. (I) γ is invariant under the inversion $x \mapsto -x$ or $y \mapsto -y$. (II) The three points $O = (0, 0)$ and $(\pm 1, 0)$ are on γ. (III) In the first quadrant, γ is described by a function as $(x, f(x))$ for $0 \leqslant x \leqslant 1$ that satisfies $f(0) = f(1) = 0$ and $f(x) > 0$ for $0 < x < 1$. (IV) For the smoothness of the curve:

$$f'(0) = \lim_{x \to +0} f'(x) > 0 \qquad \text{and} \qquad \lim_{x \to 1-0} f'(x) \to -\infty. \tag{33}$$

These properties (I)–(IV) are acceptable as those for the usual eight-shaped curves.

We look for the solution $q_1, q_2 \in \gamma$ satisfying $q_1 + q_2 + q_3 = 0$ for a given $q_3 \in \gamma$ in the cross points $\gamma \cap \gamma_{\|}$ according to theorem 1. Since the origin $O \in \gamma$, we find the trivial solution $\{q_1, q_2\} = \{O, -q_3\}$ for $q_3 \in \gamma$ in which the three points q_3, O and $-q_3$ are collinear.

See figures 4, 6 and 7. This trivial solution, however, has no physical importance since it does not conserve the angular momentum

$$\sum_{i=1,2,3} q_i \times \frac{\mathrm{d}q_i}{\mathrm{d}t} = 0 \times 0 + (-q_3) \times \frac{\mathrm{d}(-q_3)}{\mathrm{d}t} + q_3 \times \frac{\mathrm{d}q_3}{\mathrm{d}t} = 2q_3 \times \frac{\mathrm{d}q_3}{\mathrm{d}t}, \qquad (34)$$

which changes the sign at $q_3 = 0$. Moreover, these three points will go to the three-body collision at the origin when $q_3 \to O$.

On the other hand, figures 4 and 7 suggest that there is just one non-trivial solution $q_1, q_2 \in \gamma$ satisfying $q_1 + q_2 + q_3 = 0$ for a given $q_3 \in \gamma - \{0\}$. In the rest of this section, we will show that if the eight-shaped curve γ has some sufficient conditions, the non-trivial pair $\{q_1, q_2\}$ is unique, smooth and strongly monotonic. The sufficient conditions are the followings. (V) The curvature of the curve is negative, namely $f''(x) < 0$ for $0 < x < 1$. (VI) The third derivative is also negative, $f'''(x) < 0$ for $0 < x < 1$.

Before describing the next theorem, note that conditions (IV) and (V) imply that there is a unique value of $x = a_0$ with $0 < a_0 < 1$ that satisfies

$$f'(a_0) = -f'(0). \qquad (35)$$

We write the point $(a_0, f(a_0)) = p_0$.

Theorem 5. *If an eight-shaped curve γ which is invariant under inversion $x \mapsto -x$ or $y \mapsto -y$ is described in the first quadrant by a curve $(x, f(x))$ with $0 \leqslant x \leqslant 1$ that satisfies $f(0) = f(1) = 0$, $f'(0)$ is positive finite, $f'(x) \to -\infty$ for $x \to 1 - 0$ and for $0 < x < 1$*

$$f(x) > 0,\ f''(x) < 0,\ f'''(x) < 0, \qquad (36)$$

the solutions of $q_1 + q_2 + q_3 = 0$ with $q_1, q_2 \in \gamma$ for a given $q_3 \in \gamma - \{0\}$ are two pairs, trivial one $\{q_1, q_2\} = \{O, -q_3\}$ and non-trivial one $\{q_1, q_2\} = \{q, q^*\}$. For the case $q_3 = p_0 = (a_0, f(a_0))$, the trivial pair and the non-trivial pair are coincident, $\{q_1, q_2\} = \{O, -p_0\} = \{q, q^*\}$ where a_0 is the unique solution of $f'(a_0) = -f'(0), 0 < a_0 < 1$. When q_3 moves around γ, the motion $q_i(\sigma), i = 1, 2,$ of the non-trivial pair are smooth, i.e. $|\mathrm{d}q_i/\mathrm{d}\sigma| < \infty$, and strongly monotonic i.e. $\mathrm{d}q_i/\mathrm{d}\sigma \neq 0$ where σ is the curve length for q_3.

A proof of this theorem will be given in the following subsections.

As mentioned in section 3.1, the motion of non-trivial pair in this theorem is uniquely parameterized as $q_i(\sigma(t))$ by the curve length $\sigma(t)$ of q_3. Since the total area of an eight-shaped curve is zero, the constant angular momentum should be zero. Thus, unlike section 3, the equation $c = 0$ gives no information for $\mathrm{d}\sigma/\mathrm{d}t$. Although vanishing angular momentum does not determine the speed of the motion, it imposes a strong constraint on the shape of curve γ. Namely, by the three tangents theorem [5–7], three tangent lines at $q_i(\sigma)$ must meet at a point for each σ.

Then we use the energy constant assuming some potential energy V:

$$V = \sum_{i<j} U(|q_i - q_j|) \qquad (37)$$

to determine $\sigma(t)$ by the Hamiltonian H:

$$H = \frac{1}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \sum_{i=1,2,3} \left|\frac{\mathrm{d}q_i(\sigma)}{\mathrm{d}\sigma}\right|^2 + V = \text{constant}. \qquad (38)$$

This condition H = constant determines the motion completely, although this motion is not guaranteed to satisfy the equation of motion derived from this Hamiltonian. Following the argument in section 3, for eight-shaped curves, again the motion $q_i(\sigma(t))$ is determined

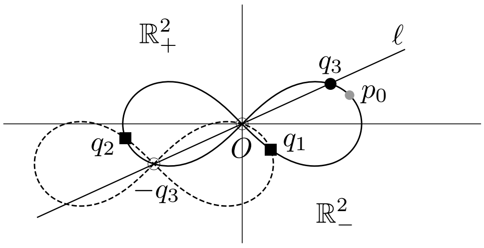

Figure 4. The curves γ (solid line) and $\gamma_\parallel$ (dashed line) for $0 < a < a_0$ where $q_3 = (a, f(a))$. Solid black and gray circles represent q_3 and p_0 respectively. Hollow circles represent the trivial solution $\{O, -q_3\}$ and the solid square represents the non-trivial solution $\{q_1, q_2\}$. The line ℓ that passes through the points O and $-q_3$ splits the plane $\mathbb{R}^2$ into $\mathbb{R}^2_+$, $\mathbb{R}^2_-$ and the line ℓ itself.

completely as a 'choreography'. Thus, a 'choreography' in an eight-shaped curve is determined by the curve γ that satisfies the three tangents theorem that ensures the angular momentum being zero and by the potential energy.

To prove theorem 5, we use theorem 1. For the curve γ, we call the right lobe R and left lobe L. Similarly, $R_\parallel$ and $L_\parallel$ for $\gamma_\parallel$. Note that $R^* = L_\parallel$ and $L^* = R_\parallel$. In the following subsections, we give a proof of theorem 5 for the cases $0 < a < a_0$, $a = a_0$ and $a_0 < a \leqslant 1$, where $q_3 = (a, f(a))$, separately.

4.1. For the case $0 < a < a_0$

See figure 4.

(1) For $R \cap R_\parallel$: it is obvious the origin $O \in R \cap R_\parallel$. Since $f''(x) < 0$ and $0 < a < a_0$, we have $f'(0) > f'(a) > f'(a_0) = -f'(0)$. Therefore, $R_\parallel$ starts the origin to inside of R. The lobe $R_\parallel$ cuts the y-axis at $(0, -2f(a))$ which is obviously outside of R. Therefore, there is at least one point $q_1 \neq O$ in $R \cap R_\parallel$. Therefore, $R \cap R_\parallel = \{O, q_1\}$ by lemma 2 in appendix A.

(2) For $L \cap L_\parallel$: by the map $q \mapsto q^*$, we get $L \cap L_\parallel = R^*_\parallel \cap R^* = \{O^*, q^*_1\} = \{-q_3, q_2\}$.

(3) For $L \cap R_\parallel = L \cap L^*$: the line ℓ connecting the origin O and $-q_3$ splits the plane $\mathbb{R}^2$ into three parts, open upper half that we write $\mathbb{R}^2_+$, open lower half $\mathbb{R}^2_-$ and the line ℓ itself. This line also splits L and L^* into three parts. We will show that $L \cap L^* \cap \mathbb{R}^2_+$ is empty. To find the number of elements of $L \cap L^* \cap \mathbb{R}^2_+$, let us consider the difference between the y component of the curve $L \cap \mathbb{R}^2_+$ and that of the curve $L^* \cap \mathbb{R}^2_+$, which is described by the following function:

$$g(x, a) = f(-x) - f(x + a) + f(a) \tag{39}$$

defined in $-a < x < 0$. In appendix B, we have shown that there is no solution of $g(x, a) = 0$ in $-a < x < 0$ for $0 < a < a_0$. Therefore, $L \cap L^* \cap \mathbb{R}^2_+$ is empty.

By the map $q \mapsto q^*$, the region $\mathbb{R}^2_+$ maps onto the region $\mathbb{R}^2_-$; therefore, $L \cap L^* \cap \mathbb{R}^2_-$ is also empty. Thus, we get $L \cap L^* = L \cap L^* \cap \ell = \{O, -q_3\}$ since L has at most two common points with any line.

(4) For $R \cap L_\parallel$: R is in the region $x \geqslant 0$, while $L_\parallel = R^*$ is in $x \leqslant -a < 0$. Therefore, $R \cap L_\parallel$ is empty.

Summarizing the results of (1)–(4), we conclude that there are one trivial pair $\{O, -q_3\} \subset \gamma \cap \gamma_\parallel$ and one non-trivial pair $\{q_1, q_2 = q^*_1\} \subset \gamma \cap \gamma_\parallel$ for the case $0 < a < a_0$. The smoothness and strong monotonicity of $q_i(\sigma)$ for $i = 1, 2$ can be proved by lemma 1 referring to figure 5.

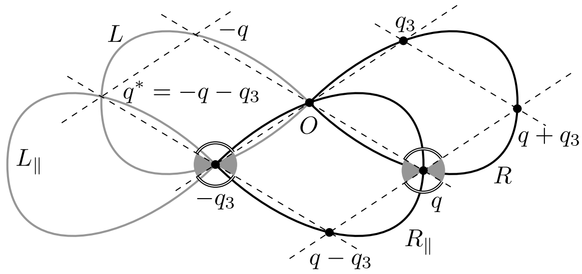

Figure 5. Lemma 1 for the eight-shaped curve for $0 < a < a_0$. We denote the cross point of R and $R_\parallel$ by q. (i) The parallelogram $\alpha q + \beta q_3, 0 \leqslant \alpha, \beta \leqslant 1$, is included inside R; thus, at q, the tangent line to the curve γ passes through the shaded area, while the tangent line to the curve $\gamma_\parallel$ at q passes in the non-shaded area because the parallelogram $\alpha q - \beta q_3, 0 \leqslant \alpha, \beta \leqslant 1$, is included inside $R_\parallel$. Therefore, the tangent lines at q to the lines γ and $\gamma_\parallel$ are distinct. (ii) The parallelogram $-\alpha q - \beta q_3, 0 \leqslant \alpha, \beta \leqslant 1$, is included inside L; thus, the tangent line to the curve γ at $-q_3$ passes through the shaded area. Therefore, the tangent line to $\gamma_\parallel$ at q and the tangent line to γ at $-q_3$ are not parallel.

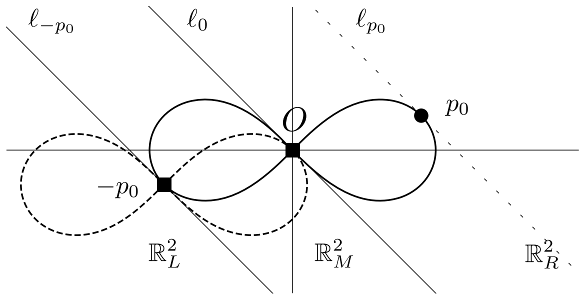

Figure 6. The case $a = a_0$. The set $\gamma \cap \gamma_\parallel = \{O, -p_0\}$. Two tangent lines ℓ_0 and ℓ_{-p_0} split the plane $\mathbb{R}^2$ into the open regions $\mathbb{R}^2_L, \mathbb{R}^2_M, \mathbb{R}^2_R$ and the lines.

4.2. *For the case $a = a_0$*

See figure 6. Let the tangent line of the curve at $p_0 = (a_0, f(a_0))$ be ℓ_{p_0}, and its parallel translation by $q \mapsto q - p_0$ and $q \mapsto q - 2p_0$ be ℓ_0 and ℓ_{-p_0}. The lines ℓ_0 and ℓ_{-p_0} are the tangent lines of the curve γ and $\gamma_\parallel$ that pass through the origin and $-p_0$ respectively. The two parallel lines ℓ_0 and ℓ_{-p_0} split $\mathbb{R}^2$ into five pieces, three open two-dimensional regions and two lines. We name the three regions from left to right $\mathbb{R}^2_L, \mathbb{R}^2_M$ and $\mathbb{R}^2_R$. Obviously, $\gamma \cap \gamma_\parallel \cap \mathbb{R}^2_L$ and $\gamma \cap \gamma_\parallel \cap \mathbb{R}^2_R$ are empty. The set $\gamma \cap \gamma_\parallel \cap \mathbb{R}^2_M$ is also empty since, by the same argument for $L \cap L^* \cap \mathbb{R}^2_+$ in the previous subsection, $g(x, a_0) = 0$ has no solution in $-a < x < 0$, which is shown in appendix B.

Therefore, $\gamma \cap \gamma_\parallel = \gamma \cap \gamma_\parallel \cap (\ell_0 \cup \ell_{-p_0}) = \{O, -p_0\}$ for $a = a_0$.

4.3. *For the case $a_0 < a \leqslant 1$*

See figure 7. Similar to section 4.2, let the tangent line of the curve at $q_3 = (a, f(a))$ be ℓ_{q_3}, and its parallel translation by $q \mapsto q - q_3$ and $q \mapsto q - 2q_3$ be ℓ_0 and ℓ_{-q_3} respectively. We use the same notations as section 4.2, $\mathbb{R}^2_L, \mathbb{R}^2_M$ and $\mathbb{R}^2_R$ for the regions. Obviously, $\gamma \cap \gamma_\parallel \cap \mathbb{R}^2_L$ and $\gamma \cap \gamma_\parallel \cap \mathbb{R}^2_R$ are empty. By the same arguments for $L \cap L^* \cap \mathbb{R}^2_+$ in section 4.1, $\gamma \cap \gamma_\parallel \cap \mathbb{R}^2_M$ is $\{q_1, q_2 = q_1^*\}$ with $q_1 = (x_0(a), f(x_0(a)))$ where $x = x_0(a)$ is the only solution of $g(x, a) = 0$ in $-a < x < 0$. See appendix B.

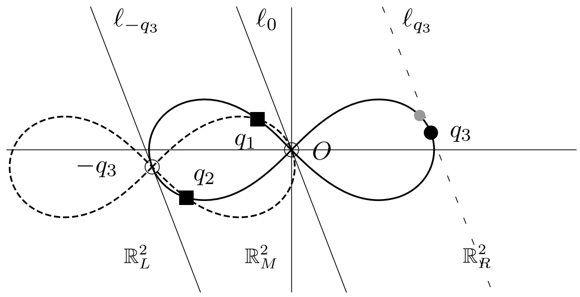

Figure 7. The case $a_0 < a \leqslant 1$. The set $\gamma \cap \gamma_\parallel = \{O, -q_3\} \cup \{q_1, q_2 = q_1^*\}$. Two tangent lines ℓ_0 and ℓ_{-q_3} split the plane $\mathbb{R}^2$ into the open regions $\mathbb{R}^2_L, \mathbb{R}^2_M, \mathbb{R}^2_R$ and the lines. The non-trivial pair $\{q_1, q_2\}$ is in the region $\mathbb{R}^2_M$.

Therefore, we conclude that the set $\gamma \cap \gamma_\parallel$ has one trivial pair $\{O, -q_3\}$ and one non-trivial pair $\{q_1, q_2 = q_1^*\}$ for the case $a_0 < a \leqslant 1$. The smoothness and strong monotonicity of $q_i(a)$ for $i = 1, 2$ are explicitly given by equation (B.19) in appendix B.

5. Summary and discussions

In this paper, we have shown that the motion of equal mass three bodies in a given curve is uniquely determined as a choreography for the following two cases. (i) Convex curves that have point symmetry with respect to the origin and non-vanishing angular momentum are given. (ii) Eight-shaped curves and the energy constant are given.

For eight-shaped curves, condition (V) in section 4, the convexity of each lobe, is numerically satisfied by the figure-eight solutions under homogeneous potential $\alpha^{-1}r^\alpha$ with $\alpha < 2$ and proved for the Newtonian potential, $-r^{-1}$, by Fujiwara and Montgomery [5]. Condition (VI) and all the other conditions are numerically satisfied by the figure-eight solution for the Newtonian potential.

Moreover, theorem 5 holds for the lemniscate curve of Bernoulli although it does not satisfy condition (VI) at a point $x_0 = \sqrt{5/32} = 0.395\,285\ldots$, i.e. $f'''(x_0) = 0$. Therefore, we know that the only possible motion of the equal mass three bodies in the lemniscate is $(x(\tau(t)), y(\tau(t)))$ with a smooth function $\tau(t)$ where

$$x(t) = \frac{\mathrm{sn}(t)}{1 + \mathrm{cn}^2(t)} \qquad y(t) = \frac{\mathrm{sn}(t)\mathrm{sn}(t)}{1 + \mathrm{cn}^2(t)}, \tag{40}$$

and, sn and cn are the Jacobian elliptic functions [6].

As for condition (VI), it seems too strong as we have seen in the proof of theorem 5. Also we note that all conditions for the theorem are geometric except for this condition. To replace this condition to more weak one and more geometric quantity is a future work.

For a general closed curve, which is not point symmetric, we can investigate the uniqueness of the equal mass three-body motion in it in the same manner. First, we investigate a non-trivial pair $\{q, q^*\}$ for all $q_3 \in \gamma$ in corollary 3. This might be lengthy and tedious as we did in this paper. However, once uniqueness and smoothness of the pair are found, the motion in such a curve is determined uniquely modulo time re-parameterization, $q_i(t) \mapsto q_i(\sigma(t))$ with the function $\sigma(t)$.

To determine the function $\sigma(t)$, we can use the constancy of the angular momentum, like Kepler did,

$$c = \sum_{i=1,2,3} q(\sigma(t)) \times \frac{\mathrm{d}q_i(\sigma(t))}{\mathrm{d}t} = \frac{\mathrm{d}\sigma}{\mathrm{d}t} \sum_{i=1,2,3} q(\sigma) \times \frac{\mathrm{d}q_i(\sigma)}{\mathrm{d}\sigma}. \tag{41}$$

The value of the constant angular momentum c is related to the total area S of the curve. If the total area S is not zero, then $c \neq 0$. Then equation (41) determines $\mathrm{d}\sigma/\mathrm{d}t$ and the motion $q_i(\sigma(t))$ are determined completely. While if $S = 0$ like an eight-shaped curve, then $c = 0$. Thus, equation (41) gives no information for $\mathrm{d}\sigma/\mathrm{d}t$. Then we can use the energy constant assuming some potential energy V to determine $\sigma(t)$ by the Hamiltonian, $H = \text{constant}$. Thus, the motion $q_i(\sigma(t))$ are determined completely.

The angular momentum and the Hamiltonian are invariant under the exchange of the bodies $1 \to 2 \to 3 \to 1$. This invariance and the uniqueness of the motion yield the three-body chase with an equal time spacing, namely 'the choreography in the given curve', $q_1(t) = q(t), q_2(t) = q(t + T/3), q_3(t) = q(t + 2T/3)$.

Since the motion $q_i(t)$ is determined uniquely, the acceleration $\mathrm{d}^2q(t)/\mathrm{d}t^2$ is also determined uniquely. Therefore, whether the equation of motion

$$\frac{\mathrm{d}^2q_i(t)}{\mathrm{d}t^2} = -\frac{\partial V}{\partial q_i} \tag{42}$$

with an appropriate potential energy V is satisfied or not is a test whether the motion is actually realized by the potential or not. However, in general, it is very hard to find the potential energy V which realizes the three-body motion in given curves.

For the figure-eight solution, the shape of the curve that corresponds to Kepler's first law is not known. The three tangents theorem [5–7] is a strict constraint for the curve and would be a clue to find it.

Finally, one may consider the general three-body problem in given curves, where the masses and orbits of three bodies are not equal, using theorem 2.

Acknowledgment

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Appendix A. Number of cross points of convex curves γ and $\gamma_\parallel$

Lemma 2. *Consider a closed convex curve γ in $\mathbb{R}^2$ that has at most two common points with any line. Then the cross points of γ and its parallel translation $\gamma_\parallel = \{q - p | q \in \gamma\}$ with $p \neq 0$ are at most 2.*

Proof. Suppose there are three distinct points a_1, a_2 and $a_3 \in \gamma \cap \gamma_\parallel$. Then, by the definition, points $a'_1 = a_1 + p, a'_2 = a_2 + p$ and $a'_3 = a_3 + p$ are also in γ. Therefore, the points $a_1, a_2, a_3, a'_1, a'_2, a'_3$ are in γ.

Take a coordinate system whose x-axis is parallel to the line $a'_i a_i$ so that the x components of a'_i are larger than those of a_i, namely $x'_i = x_i + |p|$. Then points a_i and a'_i have the same y component y_i. Rename the points so that the y components of the points are $y_1 < y_2 < y_3$. (These values are distinct, otherwise more than three points are in a line.) Take an oblique coordinates whose y-axis is parallel to the line a_1a_3. Then the components of the points are $a_1 = (x, y_1), a_2 = (x_2, y_2), a_3 = (x, y_3), a'_1 = (x + |p|, y_1), a'_2 = (x_2 + |p|, y_2), a'_3 = (x + |p|, y_3)$.

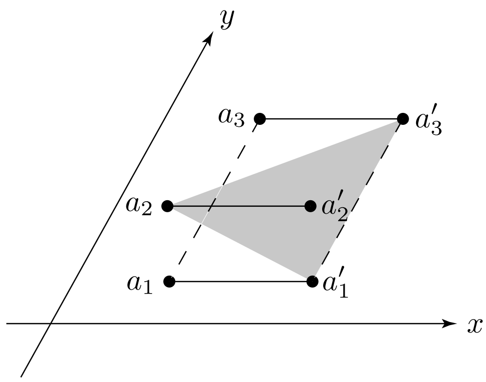

Figure A1. If the x-coordinate of a_2 is smaller than that of a_1 and a_3, then the point a_2' is inside of the triangle $a_1'a_2a_3'$.

If $x_2 < x$, then a_2' is in the triangle $a_1'a_2a_3'$. See figure A1. If $x < x_2$, then a_2 is in the triangle $a_1a_2'a_3$. Both cases contradict to the convexity of the curve γ. If $x_2 = x$, then three points a_1, a_2 and a_3 are in a line, which contradicts to the assumption of the lemma.

This contradiction comes from the assumption of the existence of the three distinct points in $\gamma \cap \gamma_{\|}$. Thus, we prove this lemma. □

Note that if the curvature of a closed convex curve is not zero almost everywhere, then the curve has at most two common points with any line. Therefore, the closed convex curve in theorem 4, and the lobe R or L of the eight-shaped curve in theorem 5 satisfy the conditions for this lemma.

Appendix B. Number of zeros for a function

Let $f(x)$ be a function defined in the region $0 \leqslant x \leqslant 1$ and satisfies the following properties:

$$f(0) = f(1) = 0, \tag{B.1}$$

$$f(x) > 0, \qquad f''(x) < 0, \qquad f'''(x) < 0 \quad \text{for} \quad 0 < x < 1, \tag{B.2}$$

and there exists a unique value a_0 in $0 < a_0 < 1$ such that

$$f'(a_0) = -f'(0) < 0. \tag{B.3}$$

We define the following function $g(x, a)$:

$$g(x, a) = f(-x) - f(x + a) + f(a), \tag{B.4}$$

in the region $0 < a \leqslant 1$, $-a \leqslant x \leqslant 0$. For a while, we consider the behavior of $g(x, a)$ for the fixed value of a. So, we simply write

$$g(x) = g(x, a) \text{ for fixed } a \tag{B.5}$$

and

$$g'(x) = \frac{\partial g(x, a)}{\partial x}. \tag{B.6}$$

We will show that the number of solutions of $g(x) = 0$ in $-a < x < 0$ is zero for $0 < a \leqslant a_0$ and 1 for $a_0 < a \leqslant 1$.

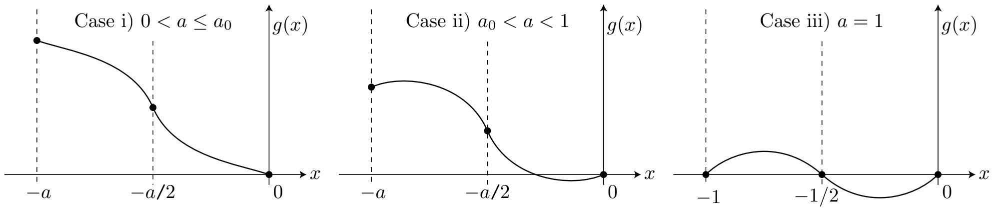

Figure B1. Schematic view of $g(x) = f(-x) - f(x+a) + f(a)$. The curve $(x, g(x))$ is point symmetric with respect to $(-a/2, g(-a/2))$, namely, $g(-x-a/2) - g(-a/2) = -(g(x-a/2) - g(-a/2))$.

Since,

$$g'''(x) = -f'''(-x) - f'''(x+a) > 0 \tag{B.7}$$

and

$$g''(-a/2) = f''(a/2) - f''(a/2) = 0, \tag{B.8}$$

we get

$$g''(x) < 0 \quad \text{for } -a \leqslant x < -a/2, \tag{B.9}$$

$$g''(x) > 0 \quad \text{for } -a/2 < x \leqslant 0. \tag{B.10}$$

Note that

$$g(-a) = 2f(a) \geqslant 0, \tag{B.11}$$

$$g(-a/2) = f(a) \geqslant 0, \tag{B.12}$$

$$g(0) = 0 \tag{B.13}$$

and

$$g'(0) = -f'(0) - f'(a) = f'(a_0) - f'(a) \tag{B.14}$$

is an increasing function of a because $f''(x) < 0$.

We split the problem into three cases by the value of a. See figure B1.

(i) The case $0 < a \leqslant a_0$: we have $g'(0) = f'(a_0) - f'(a) \leqslant 0$. Because $f(a) > 0$, we have $g(-a) > 0$, $g(-a/2) > 0$ and $g(0) = 0$. Then inequalities (B.9) and (B.10) prove that there is no solution of $g(x) = 0$ in $-a < x < 0$ in this case.

For the case $a = a_0$, we have $g(0) = g'(0) = 0$ and $g''(0) > 0$. Therefore, $g(0) = 0$ is a double root.

(ii) The case $a_0 < a < 1$: we have $g'(0) > 0$, $g(-a) > 0$, $g(-a/2) > 0$ and $g(0) = 0$. Therefore, inequalities (B.9) and (B.10) prove that there is no solution of $g(x) = 0$ in $-a < x \leqslant -a/2$ and one solution in $-a/2 < x < 0$. Let us write the zero point $x_0(a)$. Note that $g'(x)$ at $x = x_0(a)$ is negative, namely

$$g'(x_0(a)) = -f'(-x_0(a)) - f'(x_0(a) + a) < 0. \tag{B.15}$$

(iii) The case $a = 1$: we have $g(-a) = g(-a/2) = g(0) = 0$. Therefore, equations (B.9) and (B.10) prove that $x = -a/2$ is the only solution of $g(x) = 0$ in $-a < x < 0$. Inequality (B.15) is also true for this case.

Now, let us consider the behavior of the zero point of g for the range $a_0 < a \leqslant 1$. To do this, it is convenient to use the full expression $g = g(x, a)$. Then we have

$$g(x_0(a), a) = f(-x_0(a)) - f(x_0(a) + a) + f(a) = 0 \tag{B.16}$$

for all $a_0 < a \leqslant 1$. Then total derivative of this expression by a yields

$$0 = \frac{\mathrm{d}g(x_0(a), a)}{\mathrm{d}a} = \left.\frac{\mathrm{d}x_0(a)}{\mathrm{d}a}\frac{\partial g(x, a)}{\partial x} + \frac{\partial g(x, a)}{\partial a}\right|_{x=x_0(a)}. \tag{B.17}$$

Therefore,

$$\frac{\mathrm{d}x_0(a)}{\mathrm{d}a}(f'(-x_0(a)) + f'(x_0(a) + a)) = f'(a) - f'(x_0(a) + a). \tag{B.18}$$

By (B.15), we have $f'(-x_0(a)) + f'(x_0(a) + a) > 0$. While $f'(a) - f'(x_0(a) + a) < 0$ by $f''(x) < 0$. Thus, we get

$$\frac{\mathrm{d}x_0(a)}{\mathrm{d}a} < 0 \quad \text{for} \quad a_0 < a \leqslant 1, \tag{B.19}$$

namely the zero point of $g(x) = 0$ appears near the origin and moves to $-1/2$ smoothly and strongly monotonically when a increases a_0 to 1.

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